Klein-Gordon Equation using Laplacian Operator in Curved Spacetime and Scalar Field around Black Hole

Abstract

Many authors had shown the analytical or numerical solutions of Klein-Gordon Equation in Schwarzschild space-time in their papers. In these papers they use Laplace-Beltrami Operator for Klein-Gordon equation in Schwarzschild space-time. However, some solutions show the behavior near event horizon, which may not match the prediction of particle's behavior with General Relativity. Here I use second order differential operator instead of Laplace-Beltrami Operator. According to this the behavior of a scalar field matches the prediction of General Relativity and other observations related to the gravitational effect.

Aim of the Project

This project is divided into two parts. In part one I describe Klein-Gordon Equation using Laplace-Beltrami Operator and check the correctness of the numerical solution, which has been done using python programming, of the Klein-Gordon Equation and later on I introduce in a simple partial differential equation instead of using Laplace-Beltrami Operator to describe this equation. Using this new form I solve the equation in Schwarzschild Spacetime or a scalar field. In other part I have shown the coupling between gravitational field and electric field near event horizon. And the result is correct as per the theory of gravitoelectromagnetism. So I guess the new form of Klein-Gordon Equation which ive discussed in this paper is valid.

Keywords

Quantum Field Theory, General Relativity, Curved Spacetime, Blackhole Klein-Gordon Equation, Laplace-Bletrami Operator, Gravitational Field, Gauge Field, Signature of Metric, Schwarzschild Spacetime, Event Horizon, Minkowski Spacetime, Local Spacetime, Flat Spacetime, Geodesics, Affin Parameter, Polar Jets, Heavy Stars, Frobenius Method, Weak Gravitational Field.

Introduction

According to Quantum Field Theory (QFT) in curved spacetime, the equation describing a behavior of particles around blackhole is Klein-Gordon Equation (K-G Equation) in curved spacetime. In that equation Laplacian operator is described by Laplace-Beltrami Operator (L-B Operator). However, the L-B Operator is a complicated Operator, therefore a coupling of gravitational field with Gauge Field is not apparent.

History of Klein-Gordon Equation

The equation was named after the physicists Oskar Klein and Walter Gordon, who proposed that it describes relativistic electrons in 1926. This is an attempt to draw a relation between special theory of relativity and quantum mechanics. This equation correctly describes the spinless relativistic composite particles, like the pion. In 1926 the Klein–Gordon equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie. On 4 July 2012, European Organization for Nuclear Research CERN announced the discovery of the Higgs boson. Since the Higgs boson is a spin-zero particle, it is the first observed ostensibly elementary particle to be described by the Klein–Gordon equation. The Klein–Gordon equation is a relativistic wave equation, related to the Schrödinger equation. It is second-order in space and time and manifestly Lorentz-covariant. This means in the form of

$$\left(\Box + \frac{\mathbf{m}^2\mathbf{c}^2}{\hbar^2}\right)\boldsymbol{\Psi}(\boldsymbol{r},\mathbf{t}) = \mathbf{0} \qquad \textbf{(1)}$$

where $\Box$ is d'Alembert operator. $\Box = \frac{\mathbf{1}}{\mathbf{c}^2}\frac{\partial^2}{\partial \mathbf{t}^2} - \boldsymbol{\nabla}^2$.

By taking quantum chromodynamics unit ***ℏ=c=1*** This gives us

$$(\square + \boldsymbol{m^2}) = \mathbf{0}. \qquad \textbf{(2)}$$

It is a quantized version of the relativistic energy–momentum relation.

Derivation of Klein-Gordon Equation

Schrö̈dinger equation for a particle moving in a potential V(x,t) is,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \qquad \textbf{(3)}$$

Now non-Relativistic Schr**ö**dinger equation for free particle, which means there is no potential V(x,t)=0 then equation turns into

$$-\frac{\hbar^2}{2m}\nabla^2\,\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \qquad \textbf{(4)}$$

$$\frac{\hat{P}^2}{2m}\Psi = E\Psi \ \text{ as } \hat{P} = -i\hbar\nabla \qquad \textbf{(5)}$$

From these equation we get $E = i\hbar\frac{\partial}{\partial t}$ **(6)**

Relativistic momentum-energy relation

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \textbf{(7)}$$

So by replacing these operators in momentum-energy equation any one can get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Psi = -\frac{m^2c^2}{\hbar^2}\Psi = -m^2\Psi \qquad \textbf{(8)}$$

This **(6)** equation is known as Klein-Gordon Equation. Rewriting RHS using the inverse of the Minkowski metric $\eta_{\mu\nu}$, **diag($-c^2$, +1, +1, +1)** we get

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \nabla^2\Psi \qquad \textbf{(9)}$$

Thus the Klein–Gordon equation can be written in a covariant notation. ∂_μ is the 4-gradient covariant components and it is defined by,

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \overrightarrow{\nabla}\right) = \left(\frac{\partial_t}{c}, \partial_x, \partial_y, \partial_z\right) = \frac{\partial}{\partial x^\mu}$$

where $x^\mu = (ct, x, y, z) \equiv (t, x, y, z)$ **(10)**

Klein-Gordon Equation Using LB Operator

The Klein-Gordon Equation in curved spacetime using Laplace-Beltrami operator is given by,

$$\frac{1}{\sqrt{-|g|}}\partial_\mu\left(g^{\mu\nu}\sqrt{-|g|}\partial_\nu\right)\Psi = -m^2\Psi \qquad \textbf{(11)}$$

Here $g^{\mu\nu}$ is the metric tensor and $|g|$ is the determinant of metric tensor and I take $\hbar$**=c=1.** Throughout this paper, I use signature of metric of $\eta_{\mu\nu}$**(-,+,+,+)**. Laplace–Beltrami operator corresponding to the Minkowski metric with is the d'Alembertian. In Schwarzschild spacetime, this equation for the a scalar field is solved in several papers. Easy way to obtain the solution is solving numerically. Numerical solution for a radial part is given in Fig. 1. In Fig. 1, the end of left side of the graph is a place where the event horizon resides.

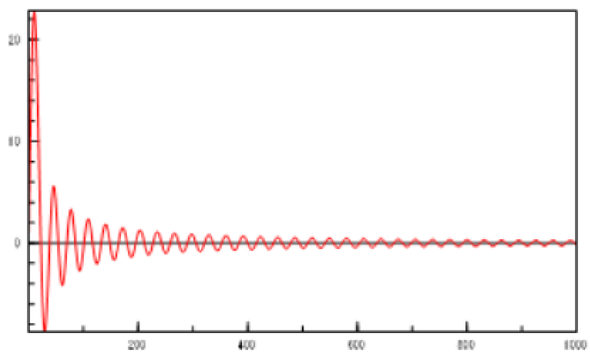

Fig. 1 the numerical solution of textbook radial K-G eq. in

Schwarzschild spacetime only outside of an event horizon is shown

According to Fig. 1, the amplitude of the scalar field monotonically increases toward the event horizon. According to the behavior of a particle predicted by General Relativity, it is said that the particle becomes faint and slow as it approaches the event horizon. However, the solution according to Fig. 1 predicts a monotonic increase of the amplitude in the scalar field, which means that as the particle approaches the event horizon, the particle gets brighter. This is contrary to the behavior of the particle according to General Relativity. Now, the question arises, "Is the solution of the scalar field using Laplace-Beltrami operator correct?"

Laplacian Operator in form of PDE

According to the principle of equivalence, local spacetime is described by Minkowski Spacetime, which means even in a curved spacetime, local spacetime becomes flat spacetime. Now, I assume that the field described in the curved spacetime is a vector field. For the observer of the vector field, each component of the vector field is measured by projecting the field to an orthogonal coordinate axes, t, x, y, and z. Although t-, x-, y-, and z-axes are a straight line, if these straight lines are transformed into the curved spacetime, these lines become geodesics in the curved spacetime. This can be easily seen as follows.

A straight line is described by the equation of motion like

$$\frac{d^2x^\mu}{d\tau^2} = 0 \qquad (12)$$

Where τ is an affine parameter. This equation describes straight lines of t-, x-, y-, and z-axes. Then, transforming the coordinate system to y^μ, one gets,

$$\frac{d^2y^\nu}{d\tau^2} = -\Gamma^\nu_{\lambda\gamma}\left(\frac{dy^\lambda}{d\tau}\right)\left(\frac{dy^\gamma}{d\tau}\right) \qquad (13)$$

Where $\Gamma^\nu_{\lambda\gamma}$ is Crystoffel symbol and defined by,

$$\Gamma^\nu_{\lambda\gamma} = \frac{\partial^2 x^\mu}{\partial y^\lambda \partial y^\gamma}\frac{\partial y^\nu}{\partial x^\mu} \qquad (14)$$

Then, the coordinate axes in Minkowski spacetime is transformed to the geodesics in the curved spacetime.

Then, in the surrounding curved spacetime, each component of the vector field along t-, x-,y-, and z- axes becomes each component along corresponding geodesics. Therefore, in describing the vector field in the curved spacetime, it is more natural to decompose the vector field along geodesics than projecting the vector field to the coordinate axes in the curved spacetime. By decomposing the vector field along geodesics corresponding to t-, x-, y-, and z-axes in the local Minkowski spacetime, the vector field is described by,

$$\Psi(t,x,y,z) = \psi_1\frac{\partial\omega^1}{\partial\tau} + \psi_2\frac{\partial\omega^2}{\partial\tau} + \psi_3\frac{\partial\omega^3}{\partial\tau} + \psi_4\frac{\partial\omega^4}{\partial\tau} \qquad (15)$$

Where $\boldsymbol{\omega^{\mu}} = (\boldsymbol{\omega^1}, \boldsymbol{\omega^2}, \boldsymbol{\omega^3}, \boldsymbol{\omega^4})$ are the geodesics in four directions in curved spacetime. That is,

$$\nabla_A \nabla_A \omega^{\mu} = \frac{d^2 \omega^{\mu}}{d\tau^2} = 0 \quad \textbf{(16)}$$

Where $\boldsymbol{\nabla_A} = (\boldsymbol{\partial_x}, \boldsymbol{\partial_{y,}} \boldsymbol{\partial_z})$ means the covariant derivative with respect to the vector field $\boldsymbol{A}$. Therefore, after some calculation, one obtains

$$\frac{d^2 \Psi}{d\tau^2} = \nabla_A \nabla_A \Psi = \frac{\partial A_x}{\partial \tau} \frac{\partial A_y}{\partial \tau} \frac{\partial^2 \psi_\mu}{\partial A_x \partial A_y} \frac{\partial \omega^\mu}{\partial \tau} = g_{A_x} g_{A_y} \frac{\partial^2 \psi_\mu}{\partial A_x \partial A_y} \frac{d\omega^\mu}{d\tau} \quad \textbf{(17)}$$

This equation says that the second order ordinary partial derivative of components of the vector field is meant to be the second order covariant derivative of the vector field. Then, Klein-Gordon equation in a curved spacetime for a scalar field, can be written using the ordinary partial derivative as,

$$\partial_\mu \partial_\nu \Psi = g_{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \Psi = -m^2 \Psi \quad \textbf{(18)}$$

In summary, Laplacian operator in the curved spacetime is described by a second order covariant derivative. Acting the covariant derivative twice to the vector field yields our desired result. However, as the vector field is described by a sum of products of component fields and tangent vectors of geodesics and the covariant deriveative derivative of a tangent vector of the geodesic becomes zero, the Laplacian operator in the curved spacetime acting on the vector field becomes a sum of products of the metric tensor, the second order partial derivative of the field components and the tangent vector of the geodesics. Therefore, looking at the equations of the field components only, the Laplacian in the curved spacetime becomes ordinary second order partial derivative rather than L-B Operator.

Scalar Field around Black Hole Under LB Operator

As shown in the previous section, the second order differential operator in Klein-Gordon equation in curved spacetime is to be interpreted as a second order ordinary partial differential operator with coefficient functions. Now, as an example, I solve the equation in Schwarzschild spacetime for a scalar field.

The inverse of the metric of Schwarzschild spacetime is given by,

$$g^{\mu\nu} = \begin{pmatrix} -(1-\frac{2M}{r})^{-1} & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r}) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \quad (19)$$

Substituting this metric in Klein-Gordon equation in curved spacetime, one gets

$$-\left(1-\frac{2M}{r}\right)^{-1}\frac{\partial^2\Psi}{\partial t^2}+\left(1-\frac{2M}{r}\right)\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}=m^2\Psi \quad (20)$$

I try to solve this equation by the method of separation of variables, which gives

$$\Psi = T(t)R(r)\Theta(\theta)\Phi(\phi) \quad (21)$$

$$\frac{\partial^2 T}{\partial t^2} = -E^2 T \quad (22)$$

$$\frac{\partial^2 R}{\partial r^2}+\left(\frac{E^2}{\left(1-\frac{2M}{r}\right)^2}+\frac{A}{r^2\left(1-\frac{2M}{r}\right)}-\frac{m^2}{\left(1-\frac{2M}{r}\right)}\right)R=0 \quad (23)$$

$$\frac{\partial^2\Theta}{\partial\theta^2}+\left(A-\frac{l^2}{\sin^2\theta}\right)\Theta=0 \quad (24)$$

$$\frac{\partial^2\Phi}{\partial\phi^2}=-l^2\Phi \quad (25)$$

The first and the last equations are easy to solve and given as a well known exponential function. Now, I try solve the second equation by a numerical method. Fig. 2 is a solution graph solving the second equation. In Fig. 2, the end of left side of the graph is a place where the event horizon resides

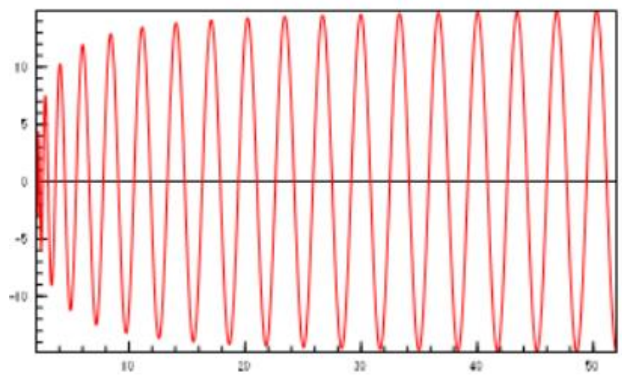

Fig. 2, numerical solution radial component of the wave function.

According to Fig. 2, reduction of the amplitude of the wave function as a particle approaches the event horizon is shown. This is a different behavior of the wave function compared with the equation using Laplace-Beltrami operator. This reduction means that the particle becomes fainter as the particle approaches the event horizon, which matches the prediction of General Relativity. Further, the amplitude of the wave function goes to constant as the particle goes to infinity. This means that there is an escaping speed to escape a gravitational attraction.

Next, the third equation which is a behavior of the wave function in a latitudinal direction in Fig. 3, both ends of the horizontal line are polar regions. The center of the horizontal line is an equator.

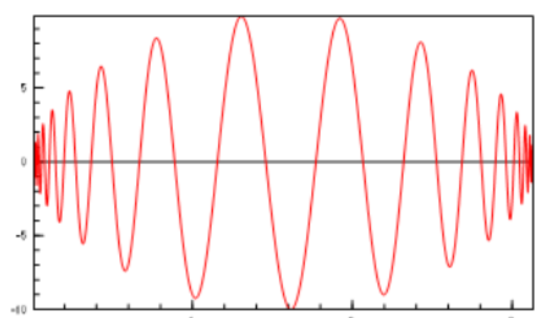

Fig. 3, numerical solution of latitudinal component

of the wave function (θ-direction)

According to Fig. 3, the wavelength of the wave function around the center is longer than other parts, which means that when particles are orbiting in the plane of the equator, particles that deviate out of the accretion disc become minor. Further, the wavelength of the wave function near the polar regions is shortened, which means that when particles deviate out of the accretion disc, the particles are accelerated toward poles and reaches very high speed at the poles. Note that in the graph of Fig. 3, plots at the poles are omitted because, at the poles, the amplitude diverges. This means that the particles are accelerated and condensed at the poles. Then, the particles may be jetted out to the polar axis directions, which possibly is an explanation of creation of the polar jets observed around heavy stars.

Next, The third equation can be transformed as,

$$\boldsymbol{z = \cos\theta} \quad \textbf{(26)}$$

$$(\mathbf{1} - \boldsymbol{z}^2)^2\boldsymbol{\Theta}'' - \boldsymbol{z}(\mathbf{1} - \boldsymbol{z}^2)\boldsymbol{\Theta}' - (\boldsymbol{A}(\mathbf{1} - \boldsymbol{z}^2) - \boldsymbol{l}^2)\boldsymbol{\Theta} = \mathbf{0} \quad \textbf{(27)}$$

The second equation can be transformed as,

$$\boldsymbol{z = \frac{2M}{r}} \quad \textbf{(28)}$$

$$\left(\frac{1}{2M}\right)^2 \left((1 - z)^2 z^6 R'' + 2(1 - z)^2 z^3 R'\right) + \left(E^2 + \frac{A}{4M^2} z^2 (1 - z) - m^2(1 - z)\right) R = 0 \quad \textbf{(29)}$$

These two equations can be solved using Frobenius Method.

Electromagnetic Potential and Gravitational Field

Now, we know that the second order differential operator in Klein-Gordon equation can be interpreted as the second order ordinary partial differential operator with coefficient functions even in curved spacetime. Next, we may consider how the gravitational field and the electromagnetic field can be coupled. As the second order derivative is only to act an ordinary partial derivative twice to the wave function, the first candidate is minimal coupling, which cannot be applied to Laplace-Beltrami operator. However, in my view, minimal coupling can be achieved by just replacing the partial differential operator with covariant derivative operator having the electromagnetic 4-potential as a connection. In order to ascertain the possibility of this method, I consider a weak gravitational field. The line element of the weak gravitational field is given by

$$-ds^2 = -(1 - 2\phi)dt^2 + (dx^2 + dy^2 + dz^2) \quad \textbf{(30)}$$

Then, the metric of the weak gravitational field is given by

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (31)$$

Therefore, the inverse of the metric of the weak gravitational field is given by

$$g^{\mu\nu} = \begin{pmatrix} -(1-2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (32)$$

If $\phi \ll 1$ is assumed. The minimal coupling is given by

$$\partial^\mu \to \partial^\mu - ieA^\mu \qquad (33)$$

In order to investigate what 4-potential is needed to eliminate the effect of the gravitational field, I equate the momentum term with the gravitational and the electromagnetic fields with the momentum term without the gravitational and the electromagnetic fields as below.

$$g_{\mu\nu}(\partial^\mu - ieA^\mu)(\partial^\nu - ieA^\nu)\Psi = \eta_{\mu\nu}\partial^\mu\partial^\nu\Psi = m^2\Psi \qquad (34)$$

Where $\eta_{\mu\nu}[\mathbf{diag}(-1,+1,+1,+1)]$ is the metric signature of a flat spacetime. Assuming a plane wave solution can be expressed as a sum of normal modes and substituting the solution into equation and assuming only scalar potential exists, we get,

$$\Psi \propto e^{i\omega t + ikx} \equiv e^{ik_\mu x^\mu} \qquad (35)$$

Where $\vec{k}$ is a special component of 4- wave number vector. $k_\mu \equiv (k_0, k_1, k_2, k_3)$. Assuming

$$\overrightarrow{k^2} = 0 \qquad \text{and} \qquad k_0 = \omega \qquad (36)$$

Let's assume $A^0 = \varphi, A^j = 0(\mathbf{for}\ j = 1,2,3)$

And ω is the frequency of the normal mode, then $\omega^2 = |p|^2 + m^2$ and $p = \hbar k$

So $\omega^2 - k^2 = m^2$ as we considered $\hbar = 1$

Now, if *e* be the charge of a particle,

$$-(\mathbf{1}+\mathbf{2}\boldsymbol{\phi})(\boldsymbol{\omega}+\boldsymbol{e\varphi})^{\mathbf{2}}+\boldsymbol{k}^{\mathbf{2}} \approx -\boldsymbol{\omega}^{\mathbf{2}}+\boldsymbol{k}^{\mathbf{2}}=-\boldsymbol{m}^{\mathbf{2}} \quad \text{as } \boldsymbol{\phi} \ll \mathbf{1} \qquad \textbf{(37)}$$

one gets

$$\boldsymbol{e}^{\mathbf{2}}\boldsymbol{\varphi}^{\mathbf{2}}+\mathbf{2}\boldsymbol{e\omega\varphi}+\boldsymbol{\omega}^{\mathbf{2}}-\frac{\boldsymbol{m}^{\mathbf{2}}}{\mathbf{1}+\mathbf{2}\boldsymbol{\phi}}=\mathbf{0} \qquad \textbf{(38)}$$

Solving this equation,

$$\boldsymbol{e\varphi}=-\boldsymbol{\omega}\pm\frac{\boldsymbol{m}}{\sqrt{\mathbf{1}+\mathbf{2}\boldsymbol{\phi}}} \qquad \textbf{(39)}$$

As $\overrightarrow{\boldsymbol{k}^{\mathbf{2}}}=\mathbf{0}\,,\ \boldsymbol{\omega}=\boldsymbol{m}$

$$\boldsymbol{e\varphi}=\boldsymbol{m}(\mathbf{1}\pm(\mathbf{1}+\mathbf{2}\boldsymbol{\phi})^{-\frac{1}{2}})$$

Then, with the assumption of small gravitational field

$$\boldsymbol{\phi} \ll \mathbf{1}\,, \qquad \boldsymbol{e\varphi} \simeq \boldsymbol{m}(-\mathbf{2}+\boldsymbol{\phi}), -\boldsymbol{m\phi} \qquad \textbf{(40)}$$

$$\vec{\boldsymbol{E}}=-\boldsymbol{\nabla\varphi}$$

$$\boldsymbol{e}\vec{\boldsymbol{E}}=\boldsymbol{m}\nabla\boldsymbol{\phi}=-\vec{\boldsymbol{F}} \qquad \textbf{(41)}$$

Where $\vec{\boldsymbol{F}}$ is the gravitational force. Therefore, we get that in order to eliminate the effect of the gravitational field, the electric force should be the same strength as the gravitational force and the directions of them are the opposite. This result is plausible. Therefore, minimal coupling is likely validated.

According to the interpretation of the second order differential operator described above, it may be possible to construct a unified theory of all four forces in nature because all gauge fields can be coupled with the gravitational field in the same manner as the electromagnetic field.

Conclusion

Starting from a question to the correctness of the solution of Klein-Gordon equation in curved spacetime, I showed that another interpretation of the second order differential operator is plausible. According to this interpretation, although the solution for a scalar field around Schwarzschild black hole needs further investigation, understanding of the dynamics of particles in curve spacetime may progress with clearer insight. Further, it may be possible to construct a unified theory of all four forces in nature.

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