On Finite Spectral Method for Axisymmetric Helmholtz Problem.

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Abstract

In this paper we investigate the numerical solution of the two-dimentional Helmholtz axy-symmetric equation via the spectral method, we reduce this problem of two-dimentional by using the cilindrical coordinates and we reduce the obtained prblem to 1-dimentional usig the orthogonal matrix, then the proposed method lood to a systeme of ordinaries equations are radher simple ..

Key-words:Axisymmetric,spectral, quadrature formula, figures illustrations. Error.

Subject classification: 33F05, 33K28.

1 Introduction

For a generic point (x, y, z) in IR^3 , we consider the Helmholtz problem in Cartesian coordinates

$$\begin{cases} -\Delta\breve{u} + k\breve{u} = \breve{f}(x, y, z) & \text{in } \Omega_3 \\ \breve{u} = \breve{g}(x, y, z) & \text{on } \partial\Omega_3 \end{cases} \tag{1}$$

where $\Delta\breve{u} = \frac{\partial^2\breve{u}}{\partial x^2} + \frac{\partial^2\breve{u}}{\partial y^2} + \frac{\partial^2\breve{u}}{\partial z^2}$ and by using the cylindrical coordinates (r, θ, z) in $IR^+ \times]-\pi, \pi] \times IR$, when the right-hand side $\breve{f}$ and the boundary $\breve{g}$ are invariant by rotation, the two-dimensional reduction reads; [5],

$$\begin{cases} -\Delta_r u + u = f(r, z) & \text{in } \Omega_2 \\ u = g & \text{on } \partial\Omega_2 \end{cases} \tag{2}$$

$$\Delta_r = -\partial_r^2 - \frac{1}{r}\partial_r - \partial_z^2 \tag{3}$$

where Ω_2 is a regular finite rectangle defined by $\Omega_2 = \Lambda \times I = [0, 1] \times [a, b]$ and ∂I the boundaries of $I = [a, b]$, $f \in L_1^2(\Omega_2)$, where :

$$L_1^2(\ \Omega_2) = \left\{ f : \ \Omega_2 \to R \text{ measurable } / \int_{\Omega_2} (f(r, z))^2 \, r dr dz < \infty \right\} \tag{4}$$

Then the problem (2) is a problem of two space variables, by using the orthogonal matrix we reduce this problem to a problem of one space variable.

In this work we construct approximate solutions to the boundary value problem (2) in the form

$$u_N(r,z)=\sum_{n=1}^{N+1}a_n(z)l_n(r) \tag{5}$$

Where the Lagrangian interpolates $l_n(r), 1\leq n\leq N+1$, are defined at the points $r_i\in\overline{I}=[0,1]\,,\,0\leq i\leq N$ and at the points $z_j\in\overline{\Lambda}=[-1,1]\,,\,0\leq j\leq N$ respectively,

where the points r_i, $0\leq j\leq N$ are the collocation points on the Gauss-Lobatto Legendre grid .

The choice of the form (5) for the solution, added to some technics gives a linear system which can be written in a matricial form as $Aa-\Gamma D^2a=\Gamma F$, where A is a square symmetric positive defined matrix and Γ is a diagonal invertible matrix and the operator $D^2=\frac{d^2}{dz^2}$. We write $a=Pv$ where P is an orthogonal matrix such that $P^{-1}\left(\Gamma^{-1}A\right)P=C$ is a diagonal matrix, then we obtain a system of $N+1$ ordinary differential equations, we can use Lagrange's method of undetermined parameters to solve for each component $v_j(z)$ of v [9], finally we conclude the expressions of functions $a_n(z)$ and for which we obtain the approximation solution.

2 Polynomials

we work in the interval $\Lambda=[0,1]$ and we use the polynomials

$$O_n(r)=(\frac{1}{n!}\frac{d^n}{dr^n}((r-r^2)^n),n\geq 0$$

occur from the Legendre polynomials with change of variable, each polynomial O_n has the degree n and in $L^2_1(\Lambda)$ satisfies the following property:

$$\int_0^1 O_n^2(r)rdr=\frac{1}{2n+1} \tag{6}$$

also we use the polynomials

$$A_n(r)=(\frac{1}{n!r}\frac{d^n}{dr^n}((r)(r-r^2)^n),n\geq 0 \tag{7}$$

has the degree $n+1$ and in $L^2_1(\Lambda)$ satisfies the following property:

$$\int_0^1 A_n^2(r)rdr=\frac{1}{2(n+1)},n\geq 0 \tag{8}$$

and we consider the differential equation:

$$h_n'(z)=-n(n+1)L_n(z) \tag{9}$$

where,

$$h_n(z)=(1-z^2)L_n'(z) \tag{10}$$

3 Variational formulation

3.1 The spaces

The pivot space of the problem (2) is the space $L_1^2(\ \Lambda)$, and the variational space is

$$H_1^1(\Lambda) = \left\{ v/\ \ \partial_r v \in L_1^2(\Lambda) \right\} \tag{11}$$

and the corresponding norms are defined respectively as,

$$\|v\|_{L_1^2(\Lambda)}^2 = \int_\Lambda v^2 r dr dz \tag{12}$$

$$\|v\|_{H_1^1(\Lambda)}^2 = \int_\Lambda ((\partial_r v)^2 + (v)^2) r dr \tag{13}$$

3.2 Continuous problem

The variational formulation of problem (2) it is written :

$$\begin{cases} \text{find } \ u \in H_1^1(\Lambda), \ \text{with } u \text{ in } H_1^1(\Lambda), \ \text{such that} \\ \forall v \in H_1^1(\Lambda), \ \ (u_{zz}, v) + a_1(u, v) = (f, v) \end{cases} \tag{14}$$

where the bilinear form $a_1(.,.)$ is given by:

$$a_1(u, v) = \int_0^1 (\partial_r u \partial_r v + uv) r dr \tag{15}$$

see[7].

4 Discrete space and form

Let us denoted by N the parameter of discretization for the problem (2), in spectral method N represent the degree of polynomials. The approximate space is essentially generated by the finite dimensional subspace of $L_1^2(\Lambda)$, $P_N(\Lambda)$ is the approximate space of the space $H_1^1(\Lambda)$, we consider also the exact quadrature formula and introduce a bilinear form a_{1N} with approach to the form a_1 and we approximate the scalar $(.,.)_N$ for $(.,.)$.

4.1 Discret problem

Firstly we observe that the products of Lagrange polynomials $l_n(r)$, $1 \leq n \leq N+1$, form a basis of $P_N(\Lambda)$, then the exact solution u of problem (2) is approached by the solution u_N belonging to $P_N(\Lambda)$, and the variational problem is:

$$\begin{cases} \text{find } u_N \in P_N(\Lambda), \ \text{s.t} \\ \forall v_N \in P_N(\Lambda), \ \ (u_{Nzz}, v_N)_N + a_{1N}(u_N, v_N) = (f_N, v_N)_N \end{cases} \tag{16}$$

where

$$a_{1N}(u_N, v_N) = \sum_{k=0}^{N} (\partial_r u_N \partial_r v_N + u_N v_N)(r_k, z)\, w_k$$

where r_k, w_k, $0 \leq k \leq N$ are defined in propositions 1 .

4.2 Existence and uniqueness of solution

4.2.1 Weighted quadrature formula

Proposition 1 *There exists a unique set of* $N+1$ *nodes* r_j, $1 \leq j \leq N+1$ *in* I *, there exists* $N+1$ *positive weights* w_j *,* $1 \leq j \leq N+1$*, such that the following exactness property holds:*

$$\forall \varphi \in P_{2N-1}(I) \ , \ \int_0^1 \varphi(r) r dr = \sum_{j=0}^{N} \varphi(r_j) w_j \tag{17}$$

where r_j *,* $1 \leq j \leq N+1$ *are the roots of the polynomial* $t_N(r) = (r - r^2) A'_N(r)$ *where* $A'_N(r) = \frac{d}{dr} A_N(r)$ *(**??**) and the weights are given by:*

$$\omega_{N+1} = \frac{1}{N(N+1)}, \ \omega_1 = \frac{\omega_{N+1}}{(N+1)^2} \tag{18}$$

$$\omega_j = \frac{4}{N(N+2) A_N^2(r_j)} \qquad 2 \leq j \leq N+1 \tag{19}$$

Proof. The proof is similar to the proof in [5]. ∎

Proposition 2 *The polynomial* q_{2N-1} *with degree* $2N-1$ *has the form*

$$q_{2N-1}(r) = (r - r^2)^2 A'^2_{N-1}(r) + \alpha(N)(r - r^2) A'^2_N(r) \tag{20}$$

$$\text{where } \alpha(N) = \frac{N+1}{2N(4N^2 - 1)}$$

Lemma 3 *The polynomials*

$$t_{N-1} = (r - r^2) A'_{N-1}(r) \in P_N(\Lambda) \tag{21}$$

verify the double inequality:

$$\|t_{N-1}\|^2_{L_1^2(\Lambda)} \leq (t_{N-1}, \ t_{N-1})_N \leq \frac{3}{2} \|t_{N-1}\|^2_{L_1^2(\Lambda)} \tag{22}$$

where $P_N(\Lambda)$ *is the space of polynomials with degree* $\leq N$ *on* Λ*.*

Proof. Using (21) we find,

$$I_1 = \int_0^1 t_{N-1}^2(r) r dr = \frac{(N^2-1)(N^2+2)}{N(4N^2-1)} \tag{23}$$

$$I_2 = \int_0^1 \left(r - r^2\right) A'^2_N(r) r dr = \frac{4N(N+2)}{2(N+1)} \tag{24}$$

$$\int_0^1 q_{2N-1}(r) r dr = I_1 + \alpha(N) I_2 \tag{25}$$

using the exact quadrature formula then

$$I_3 = \int_0^1 q_{2N-1}(r) r dr = \sum_{i=0}^{N} t_{N-1}^2(r_i) \omega_i$$

we conclude that,

$$\|t_{N-1}\|^2_{L^2_1(I)} \leq (t_{N-1}, t_{N-1})_N \tag{26}$$

let define γ s.t $\gamma\frac{I_1}{I_3} \geq 1$ that's give

$$\gamma > 1$$

then we can take,

$$\gamma = \frac{3}{2}$$

finally we find $I_3 \leq \gamma I_1$ and then (25) and (26) give the desired result (22) ■

Proposition 4 *The bilinear form $a_{1N}(\cdot,\cdot)$ satisfy the following properties of continuity:*

$$\forall v_N \in P_N(\Lambda), \forall u_N \in P_N(\Lambda), \ \ |a_{1N}(u_N, v_N)| \leq \frac{3}{2}\|u_N\|_{H^1_1(\Lambda)} \cdot \|v_N\|_{H^1_1(\Lambda)} \tag{27}$$

and ellipticity:

$$\forall u_N \in P_N(\Lambda), \ \ |a_{1N}(u_N, u_N)| \geq \|u_N\|^2_{h^1_1(\Lambda)} \tag{28}$$

5 Method of solution and numerical implementation

The problem (16) is equivalent to,

$$\begin{cases} \sum\limits_{n=1}^{N-1}(\sum\limits_{k=1}^{N+1}(-l''_n(r_k) - l'_n(r_k)/r_k + l_n(r_{kl}))l_m(r_k)\omega_k a_n(z) - l_n(r_k)l_m(r_k)\omega_k a''_n(z)) = \sum\limits_{n=1}^{N-1} f_n(t)l_n(r_k) & \text{in } \Lambda \\ u_N(r,z) = g(z) & \text{on } \partial I = \partial([a,b]) \\ f(r,t) = \sum\limits_{n=1}^{N+1} f_n(r_n,t)l_n(r), & , \ r_n \in \sum_{N+1} \end{cases} \tag{29}$$

when m vary from 1 to $N-1$, we obtain a linear system, then we can write this system in a matricial form:

$$Aa - \Gamma D^2 a = \Gamma F \tag{30}$$

where A is a symmetric matrix positive defined with order $N-1$, its elements have the form:

$$\alpha_{mn} = \sum_{k=1}^{N+1}(-l''_n(r_k) - l'_n(r_k)/r_k + l_n(r_k))l_m(r_k)\omega_k, \ n = \overline{1, N-1}\,\}, m = \overline{1, N-1}$$

Γ is a diagonal invertible matrix its elements are define as:

$$\gamma_{mn} = \begin{cases} \omega_m & , n = m \\ 0, & n \neq m \end{cases} \quad , \ m,n = \overline{1, N-1}$$

F is a known vector where:

$$F = (f_1(z), f_2(z), f_3(z), ..., f_{N-2}(z), f_{N-1}(z))^T$$

and the vector a is an unknown vector where

$$a = (a_1(z), a_2(z), a_3(z), ..., a_{N-2}(z), a_{N-1}(z))^T$$

the operator,

$$D^2 = \frac{d^2}{dz^2}$$

multiplying (30) by the invertible matrix Γ^{-1} of Γ then we find

$$\Gamma^{-1}Aa - D^2 a = F \tag{31}$$

the matrix $\Gamma^{-1}A$ has positive eigenvalues and there exists an orthogonal invertible matrix P such that,

$$P^{-1}\left(\Gamma^{-1}A\right)P = C$$

where C is a diagonal matrix, the elements of the diagonal are the eigenvalues $\alpha_i, i = \overline{1, N+1}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$a = Pv$$

then the system (31) becomes

$$(\Gamma^{-1}A)Pv - PD^2 v = F \tag{32}$$

multiplying (32) by the matrix P^{-1} we obtain,

$$Cv - D^2 v = P^{-1}F \tag{33}$$

The matricial form (33) has $N-1$ linear equations defined as

$$\begin{aligned} v_i''(t) - \alpha_i v_i(t) &= h_i(t) \\ \text{where} \qquad h_i(t) &= -\sum_{j=1}^{N-1} p^{-1}(i,j)\, F_j(t) \ , \ 1 \leq i \leq N-1 \end{aligned} \tag{34}$$

$p^{-1}(i,j)$ are the elements of the inverse matrix P^{-1}. To solve the equations (34) we use Lagrange's method of undetermined parameters [9], we may write the solution in the closed form :

$$v_i(z) = -\frac{1}{\mu_i}\int_{-1}^{z} \sinh(\mu_i(z-s))h_i(s)ds + c_i e^{-\mu_i t} + d_i e^{\mu_i t}, \quad \mu_i = (\alpha_i)^{(1/2)} \tag{35}$$

where d_i and c_i are constants to be determined, using the boundary conditions then (35) may be written in the following form:

$$v_i(t) = -\frac{1}{\mu_i}\int_{-1}^{z} \sinh(\mu_i(z-s))h_i(s)ds - \frac{1}{\mu_i}\frac{\int_{-1}^{1}\sinh(\mu_i(1-s))h_i(s)ds}{\sinh(2\mu_i)}\sinh(\mu_i(z+1) \tag{36}$$

where

$$\begin{cases} v_i(-1) = \sum\limits_{j=1}^{N-1} p_{ni}^{-1} u(r_j,-1) \\ v_i(1) = \sum\limits_{j=1}^{N-1} p_{ni}^{-1} u(r_j,1) \end{cases} \quad i = \overline{1, N-1}$$

p_{ni}^{-1}, $n,i=\overline{1,N+1}$ are the entries of the inverse matrix P^{-1} . Finally we obtain the functions,

$$a_n(t)=\sum_{i=1}^{N-1}p_{ni}(-\frac{1}{\mu_i}\int_{-1}^{z}\sinh(\mu_i(z-s))h_i(s)ds-\frac{1}{\mu_i}\frac{\int_{-1}^{1}\sinh(\mu_i(1-s))h_i(s)ds}{\sinh(2\mu_i)}\sinh(\mu_i(z+1)),n=\overline{1,N-1}$$

where p_{nj} , $1\leq n,j\leq N-1$ are the elements of the matrix P, using (5) and (36) we obtain the approximate solution

$$\begin{aligned}u_N(r,z) &= \sum_{n=1}^{N-1}\sum_{i=1}^{N-1}p_{ni}\frac{1}{\mu_i}\left(-\frac{1}{\mu_i}\int_{-1}^{z}\sinh(\mu_i(z-s))h_i(s)ds+\right.\\ &\left.-\frac{1}{\mu_i}\frac{\int_{-1}^{1}\sinh(\mu_i(1-s))h_i(s)ds}{\sinh(2\mu_i)}\sinh(\mu_i(z+1))\right)l_n(r)\end{aligned}$$

5.1 Error estimate

Definition 5 *The polynomial space $P_N(\Omega_2)$ dense in the space of continuous functions on Ω_2 hence in $H_1^1(\Omega_2)$ then any function $u\in H_1^1(\Omega_2)$ admits the expansion*

$$u(r,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha(n,m)A_n(r)h_m(z)$$

and using (21) and (9) we can write

$$u(r,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\gamma(n,m)A_n(r)L_m(z) \tag{37}$$

Proposition 6 *The following estimate holds between the exact solution u in $H_1^1(\Omega_2)$ and the approximation solution $u_N\in P_N(\Omega_2)$ verify,*

$$\|u-u_N\|_{L_1^2(\Omega_2)}\leq c_1N^{-2}\|f-f_N\|_{L_1^2(\Omega_2)},\ c \text{ is a real positive constant} \tag{38}$$

Proof. Using the ellipticity condition (28) we can write,

$$\begin{aligned}\|u-u_N\|_{H_1^1(\Omega_2)}^2 &\leq a_1(u-u_N,u-u_N)=(f-f_N,u-u_N)\\ &= c_1\int_{\Omega_2}(\ (f-f_N)(u-u_N))rdr\ ,\ c_1\text{ is a real positive constant}\end{aligned} \tag{39}$$

using (**??**) and (21) and the orthogonality properties we can write

$$\frac{1}{N^2}\leq\int_{\Omega_2}h_N'^2(z)t_N^2rdrdz$$

also

$$\frac{1}{N^2}\leq\int_{\Omega_2}h_N^2(z)t_N'^2rdrdz$$

using (13) we can write

$$\|u-u_N\|^2_{H^1_1(\Omega_2)} = \|\partial_z(u-u_N)\|^2_{L^2_1(\Omega_2)} + \|\partial_r(u-u_N)\|^2_{L^2_1(\Omega_2)} + \|u-u_N)\|^2_{L^2_1(\Omega_2)} \quad (40)$$

$$\frac{1}{N^2}\|u-u_N\|^2_{L^1_1(\Omega_2)} \leq c_1 \int_{\Omega_2} (\ (f-f_N)\,(u-u_N))rdr \quad (41)$$

$$\leq c_1 \|u-u_N\|_{L^1_1(\Omega_2)} \|f-f_N\|_{L^1_1(\Omega_2)} \quad (42)$$

then we find,

$$\|u-u_N\|_{L^2_1(\Omega_2)} \leq c_1 N^{-2} \|f-f_N\|_{L^2_1(\Omega_2)}$$

this is the desired result. ■

5.2 Figure illustration

The figures 1 and 2 present the true and the approximate solution u and u_N respectively, figures 3 and 4 present the true and the approximate function $b_k(z) = u(r,z)$, $a_k(z) = u(r,z)$, $r = r(k)$, $k = \overline{0,N}$ respectively, figures 5 and 6 present the true and the approximate function $b_k(r) = u(r,z)$, $a_k(r) = u(r,z)$, $z = z(k)$, $k = \overline{0,N}$ respectively, these plots occur when $N = 10$ and the test function is : $u(r,t) = r\cos\left(\frac{\pi}{2}r\right)\sin(\pi z)$.

These plots are occurred when $N = 10$ and $(r,t) \in [0,1]\times[-1,1]$

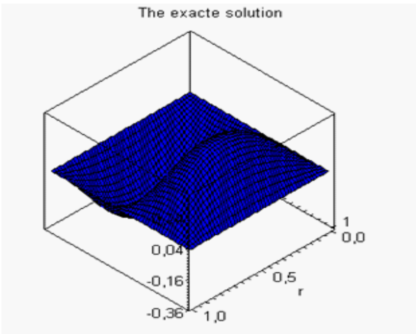

Figure 1

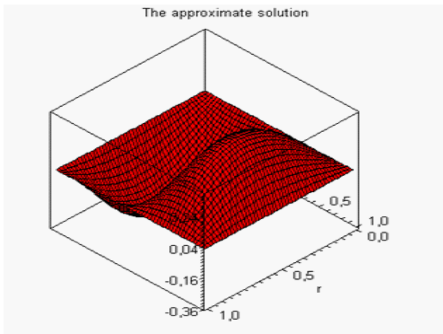

Figure 2

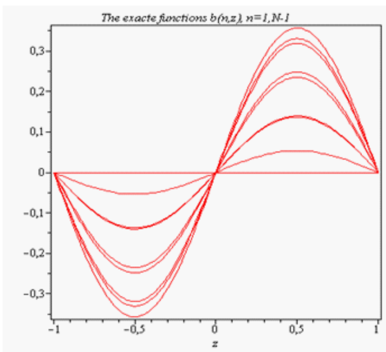

Figure 3

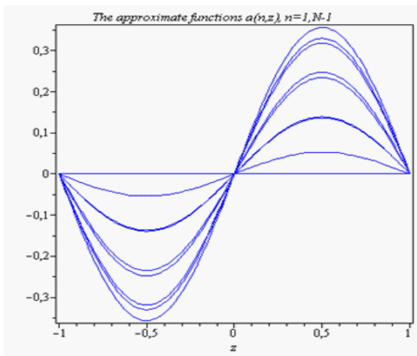

Figure 4

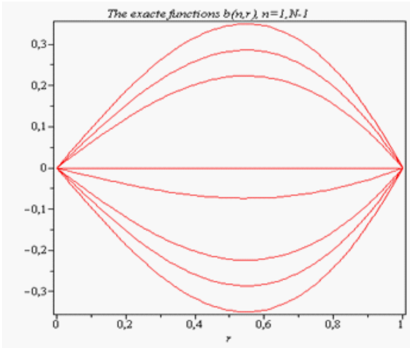

Figure 5

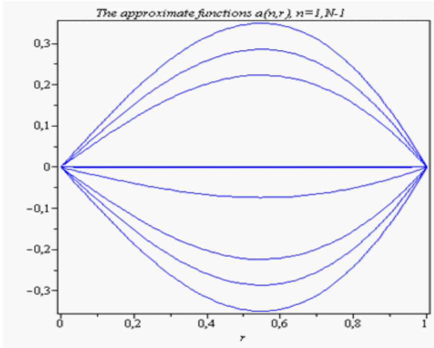

Figure 6

The behavior of the error n vary from 2 to 10

N	2	3	4	5
$Error(N)$	0.653×10^{-1}	0.447×10^{-2}	0.174×10^{-3}	0.148×10^{-4}
N	6	7	8	9
$Error(N)$	0.608×10^{-5}	0.159×10^{-8}	0.747×10^{-9}	0.185×10^{-3}

The graphs made by Maple see [6].

Conclusion 7 *In this paper the order of the matrix from* $(N-1)^2$ *to* $(N-1)$ *, by using the technics of the orthogonal matrix , then the solution presented in this method is comparably with others methods good ,and give higher accuracy than the finite difference method and spectral method with two variables.*

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